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A MATHEMATICAL THEORY FOR VARIABLE-COEFFICIENT LANCHESTER-TYPE EQUATIONS OF 'MODERN WARFARE'

by

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and

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A mathematical theory is developed for the analytic solution to deterministic Lanchester-type "square-law" attrition equations for combat between two homogeneous forces with temporal variations in system effectiveness (as expressed by the Lanchester attrition-rate coefficient). Particular attention is given to solution in terms of tabulated functions. For this purpose Lanchester functions are introduced and their mathematical properties that facilitate solution given. The above theory is applied

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20. ABSTRACT

to the following cases: (1) lethality of each side's fire proportional to a power of time, and (2) lethality of each side's fire linear with time but a nonconstant ratio of these. By considering the force-ratio equation, the classical Lanchester square law is generalized to variable-coefficient cases in which it provides a "local" condition of "winning."

SUMMARY

This paper develops a mathematical theory for the analytic solution to deterministic Lanchester-type "square-law" attrition equations for combat between two homogeneous forces with temporal variations in system effectiveness (as expressed by the Lanchester attrition-rate coefficient). Previous results of one of the authors only took a convenient form under rather restrictive conditions. Particular attention is given to solutions in terms of tabulated functions. For this purpose Lanchester functions are introduced and their mathematical properties that facilitate solution given. Tabulations of these functions would lead to analysts being able to generate numerical solutions to variable-coefficient Lanchester-type equations with somewhat the same facility as for the constant-coefficient case. It is shown that the solution to such variable-coefficient equations may be expressed in terms of four such Lanchester functions (two sets of "fundamental systems" of solutions) in a form which is a generalization of the well-known constant-coefficient solution. This allows numerical results for a single battle to generate numerical solutions for an entire family of battles and hence facilitate parametric analyses in systems analysis studies. Attention is also given to the determination of the form of attrition rate coefficients which leads to simplification of results. The above theory is applied to the following cases: (1) lethality of each side's fire proportional to a power of time, and (2) lethality of each side's fire linear with time but a nonconstant ratio of these. The latter case models the constant speed approach between forces whose weapons have different effective ranges, and in all cases the opening range of battle may be less than the effective range of weapon systems. By considering the force-ratio equation, the classical Lanchester square law is generalized to variable coefficient cases in which it provides a "local" condition of "winning."

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A MATHEMATICAL THEORY FOR VARIABLE-COEFFICIENT LANCHESTER-TYPE EQUATIONS OF 'MODERN WARFARE'

by

James G. Taylor and Gerald G. Brown

1. Introduction.

Deterministic (in the sense that the outcome is determined with certainty by initial conditions) Lanchester-type equations of warfare (see [37], [43]) are of value for identifying trends in weapon system analysis or force structuring studies because of their computational convenience, even though combat between two opposing military forces is indeed a complex random process (see Note 1). The work of S. Bonder [6], [8], [11] on methodology for the evaluation of military systems (in particular, mobile systems such as tanks, mechanized infantry combat vehicles, etc.) provides a motivation for interest in variable-coefficient deterministic Lanchester-type combat formulations and having analytic solutions to these available (see Note 2) (especially as this facilitates parametric analysis (see [8])). Bonder [7] has pointed out that (at least for the case of mobile weapon systems) the validity of the assumption of constant attrition-rate coefficients is open to question. [Indeed, Bonder's work [6], [8] has stimulated the interest of the authors on this subject.] Additionally, analytic results for such variable-coefficient formulations have also proven useful in the study of the optimal control of deterministic Lanchester-type attrition processes [36], [38], [39].

In several previous papers one of the authors has given analytic solutions for a few special cases of variable-coefficient Lanchester-type "square-law" attrition equations for combat between two homogeneous forces. In [34] (in which Taylor inadvertently rediscovered some results apparently first observed by B. O. Koopman [35] (see pp. 65-67 of [29])) it was shown that the solution to such equations is no more complicated than that for constant coefficients when the ratio of attrition-rate coefficients is constant (see also [11]). In [37] more general results were given via successive approximations and applied to power attrition-rate coefficients (both for a case which may be thought to apply for two weapon systems with the same maximum effective range and for one with coefficients linear in time but reflecting, for example, different effective ranges). The infinite series solutions given in [37] only apply, moreover, to the restrictive case in which the lethality of at least one side's fire is initially zero. One interpretation of this is that the battle must begin at the minimum of the maximum effective ranges of the two systems. The new theoretical results of the paper at hand allow these previous results to be extended to more general cases of interest.

Consideration is given to solution in terms of tabulated functions. Since tabulations of such functions for the cases at hand do not exist, we introduce Lanchester functions that should prove useful in this respect and give their mathematical properties that facilitate solution. Additionally, we determine what assumptions about the attrition-rate coefficients lead to simplification of analytic results.

The organization of this paper is as follows. We first consider the general model for which analytic results are developed and then discuss the concept of solution in terms of tabulated functions. The main results of this paper are the development of

a mathematical theory of Lanchester-type equations for a "square-law" attrition process for combat between two homogeneous forces. These general results are then applied to some special types of attrition-rate coefficients of interest, and new mathematical functions, Lanchester functions, which could be tabulated in the future are proposed. Some numerical examples are given. Then, we show how the classical Lanchester square law generalizes to such variable-coefficient formulations. Finally, we discuss extensions, the significance, and applications of our results.

2. Lanchester-Type Equations of Modern Warfare.

In 1914 F. W. Lanchester, an English aeronautical engineer who lived from 1868 to 1946 (see [27]), in order to provide insight into the dynamics of combat under "modern conditions" and justify the principle of concentration (see Note 3), hypothesized that combat between two opposing forces could be modelled by [25] (see Note 4)

$$\begin{aligned} dx/dt &= -a(t)y \qquad \text{with } x(t=0) = x_0, \\ dy/dt &= -b(t)x \qquad \text{with } y(t=0) = y_0, \end{aligned} \tag{1}$$

where $x(t)$ and $y(t)$ denote the numbers of X and Y combatants and $a(t)$ and $b(t)$ are (today) called Lanchester attrition-rate coefficients. These coefficients represent the lethality of each side's fire.

We shall refer to the equations (1) as Lanchester's equations of modern warfare. Two sets of circumstances under which these equations have been hypothesized to apply are as follows:

(a) both sides use aimed fire and target acquisition times are constant, independent of the force levels (a special case is when they are negligible) [43],

(b) both sides use area fire and a constant density defense [12].

A more complete discussion of these hypotheses (for constant lethality of fires) is to be found in the above referenced original papers. Other factors may be included in the equations and other differential equation models of combat attrition may be referred to as Lanchester-type equations, but we will not consider these here (see [16], [37]).

In Lanchester's original work [25], the attrition-rate coefficients were assumed to be constant. When either this is true or their ratio is constant, i.e. $a(t)/b(t) = k_a/k_b$, the classical Lanchester square law results

$$k_b(x_0^2-x^2(t)) = k_a(y_0^2-y^2(t)), \tag{2}$$

which has the important implication that a side can significantly reduce its casualties by initially committing more forces to battle. As we show below by consideration of the force-ratio equation, a generalization of (2) holds even for the general case of variable coefficients in which $a(t)/b(t)$ is not constant. Thus, it seems appropriate to refer to (1) as the equations for a "square-law" attrition process (see also [36], [39]).

Two significant developments in the Lanchester theory of combat during the 1960's were (a) the development of methodology for the prediction of Lanchester attrition-rate coefficients from weapon system performance data by S. Bonder [7], [9] and others [3], [11], and (b) G. Clark's development of methodology for the (maximum likelihood) estimation of such coefficients from Monte Carlo simulation data [13]. Both these developments (and others [31], [32]) have facilitated the application for defense planning studies of models such as (1) and its generalizations to combat between heterogeneous forces (see [11]).

Very recently, F. Grubbs and J. Shuford [19] have applied concepts from reliability theory to develop a new probabilistic formulation for Lanchester combat theory. They consider the random combat process to be modelled by the expected fraction of survivors, i.e. x/x_0 and y/y_0, being functionally related to various forms of probability distributions for the random time to kill. Although they consider mainly cases in which friendly losses are <u>not</u> related to the enemy force level, they do set up the following model (which is not solved)

$$d[(x_0-x)/x_0]/dt = f(t)y/y_0 \quad \text{and} \quad d[(y_0-y)/y_0]/dt = g(t)x/x_0, \tag{3}$$

where $f(t)$ and $g(t)$ are the time-to-kill probability density functions. They suggest that the Weibull distribution is a convenient point of departure for the latter. Our general results here may be applied to the formulation (3) suggested by these authors [19]. It should be pointed out that a significant advantage of the fomulation (3) over the usual continuous parameter Markov chain formulation is one of computational convenience (see [13]).

The Lanchester attrition-rate coefficients in (1) depend on a number of variables such as firing doctrine, firing rate, rate of target acquisition, force separation, tactical posture of targets, etc. (see [6] or pp. 18-26 and pp. 81-114 of [11] for attrition-rate coefficient prediction methodology). Bonder [6] (see also [11]) has considered a number of forms for attrition-rate coefficients based on examination of data for some representative weapon systems. Motivated by this work, we will consider the following coefficients in the paper at hand

$$\text{(I)} \quad a(t) = k_a(t+C)^m \quad \text{and} \quad b(t) = k_b(t+C)^n \quad \text{with} \quad C \geq 0, \tag{4}$$

$$\text{(II)} \quad a(t) = k_a(t+C) \quad \text{and} \quad b(t) = k_b(t+C+A) \quad \text{with} \quad A,C \geq 0. \tag{5}$$

Some situations which may be modelled with these coefficients are discussed below. [Additionally, for m and n negative (for $m,n < -1$ we require $C > 0$) equations (1) with (4) may be considered to model, for example, an infantry "fire fight" in which the combatants "take cover" so that the lethality of fires decreases with time.]

The results given in [37] were for the case in which $C = 0$ (see equations (4) and (5) above). For $C > 0$ the methods of [37] (both successive approximations and infinite series) have not been mathematically tractable. However, the theoretical results of the paper at hand considerably simplify the situation.

3. Tabulated Functions Arising in Military Operations Research.

It seems appropriate to discuss two points: (a) the importance of solution in terms of "tabulated functions," and (b) why (1) with either (4) or (5) does not lead (in general) to previously tabulated functions. This should help provide perspective on the contributions of the paper at hand.

Let us consider the case in which $a(t)$ and $b(t)$ are constants, e.g. $a(t) = a = \text{constant}$. Then, we say that we have "solved" for the X force level when we obtain $x(t) = x_0 \cosh\sqrt{ab}\, t - y_0\sqrt{a/b} \sinh\sqrt{ab}\, t$. This result is useful because the solution is expressed in a form (i.e. in terms of two functions of a single argument and tabulations of these functions are available (see, for example, [1])) that allows us to conveniently generate numerical results. Thus, we can generate a numerical value for the X force level for given values of the parameters x_0, y_0, a, b, and t by computing the value of the argument, i.e. $\sqrt{ab}\, t$, extracting (possibly using interpolation) the corresponding values of the hyperbolic functions from tables, and some simple algebraic manipulation. [Although a representation of such a transcendental function as an infinite series may be useful for generating such tables, it is clear that the availability of such tables is highly desirable.] One goal of our research has been to reduce the solution of (1) (with coefficients like (4) or (5)) to as close as possible to the above situation.

In trying to develop a solution for (1) with (4) or (5), the authors have, in general, not been able to express results in terms of previously tabulated functions (see below). This is not surprising when one recalls that a significant number of the so called "special functions" (see [1]) arise in the solution of equations of

mathematical physics in various coordinate systems (see Chapter 5 of [28]). Moreover, the X force-level equation (6) with coefficients (5) appears to be one not previously encountered (see Note 5).

Thus, one contribution of this paper is to point out that essentially new special functions are involved in solving the above variable-coefficient Lanchester-type equations and that tabulations of the Lanchester functions introduced below would facilitate this. It should be pointed out that a number of "special functions" and various tabulations of these have previously arisen in target coverage problems of military operations research (see, for example, [17]).

4. A Mathematical Theory of Lanchester-Type Equations of Modern Warfare.

The Lanchester-type equations (1) yield the X force-level equation

$$\frac{d^2x}{dt^2} - \left\{\frac{1}{a(t)}\frac{da}{dt}\right\}\frac{dx}{dt} - a(t)b(t)x = 0, \tag{6}$$

with initial conditions

$$x(t=0) = x_0, \qquad \text{and} \qquad \left\{\frac{1}{a(t)}\frac{dx}{dt}\right\}_{t=0} = -y_0.$$

The solution of (2) is given by

$$x(t) = C_1x_1(t) + C_2x_2(t), \tag{7}$$

where $\{x_1(t),x_2(t)\}$ denotes a fundamental system of solutions (see p. 119 of [23]). These functions can be chosen to satisfy

$$\frac{dx_1}{dt} = ka(t)y_2, \qquad \text{and} \qquad \frac{dx_2}{dt} = ka(t)y_1, \tag{8}$$

where $\{y_1(t),y_2(t)\}$ denotes a fundamental system of solutions to the Y force-level equation. The complementary Y-functions $y_1(t)$ and $y_2(t)$ consequently satisfy equations similar to (8).

In (8) above, the constant k is completely arbitrary and depends on how the fundamental X- and Y-functions are defined. It is convenient to define these so that they reduce to well-known elementary (transcendental) functions in special cases. One such special case is when

$$a(t) = k_a h(t), \qquad \text{and} \qquad b(t) = k_b h(t). \tag{9}$$

In this case it is natural to take, for example,

$$x_1(t) = \cosh\,\theta(t), \qquad \text{and} \qquad y_2(t) = \sinh\,\theta(t), \tag{10}$$

where $\theta(t) = \sqrt{k_a k_b}\int_0^t h(s)ds + \theta(t=0)$. It follows that $k = \sqrt{k_b/k_a}$.

The complementary X- and Y-functions possess the rather remarkable property that (independently of the determination of k)

$$x_1(t)y_1(t) - x_2(t)y_2(t) = 1 \qquad \forall t. \tag{11}$$

This property (11) is essential for determining the constants C_1 and C_2 of (3) in the general case in which $x_1(t=0) \neq 0$ and $x_2(t=0) \neq 0$. The properties of the general Lanchester functions are summarized in Table I. In general, the function $x_1(t)$ and $y_1(t)$ are sort of like the hyperbolic cosine (with the appropriate argument), while $x_2(t)$ and $y_2(t)$ are sort of like the hyperbolic sine (see (13) below).

Let us now sketch the proof of (11). By Abel's identity (see p. 75 of [23]), the Wronskian of x_1, x_2, denoted as $W(x_1,x_2)$, satisfies

$$W(x_1,x_2) = \begin{vmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{vmatrix} = \alpha\,\exp\{-\int\left[-\frac{1}{a(t)}\frac{da}{dt}\right]dt\},$$

whence for all t

$$\alpha = k\{x_1(t)y_1(t) - x_2(t)y_2(t)\}, \tag{12}$$

where α is a constant to be subsequently determined. It will be convenient to choose the above general Lanchester functions so that at $t = t_0$ we have

$$x_1(t=t_0) = y_1(t=t_0) = 1, \qquad \text{and} \qquad x_2(t=t_0) = y_2(t=t_0) = 0. \tag{13}$$

It follows that $\alpha = k$, and the proof of (11) is complete.

TABLE I.

Properties of the General Lanchester Functions x_1, x_2, y_1, y_2.

1. $$\frac{dx_1}{dt} = \sqrt{\frac{k_b}{k_a}}\, a(t)y_2$$

2. $$\frac{dx_2}{dt} = \sqrt{\frac{k_b}{k_a}}\, a(t)y_1$$

3. $$\frac{dy_1}{dt} = \sqrt{\frac{k_a}{k_b}}\, b(t)x_2$$

4. $$\frac{dy_2}{dt} = \sqrt{\frac{k_a}{k_b}}\, b(t)x_1$$

5. $$x_1(t)y_1(t) - x_2(t)y_2(t) = 1 \qquad \forall t$$

The general Lanchester functions x_1, x_2, y_1, and y_2 may be determined by either successive approximations or the method of Frobenious (see [37]). The time t_0 is chosen so that the general Lanchester functions take a convenient form. Examples of this are given below.

The constants C_1 and C_2 in (7) are determined by the initial conditions for (6), and thus via (8) and (11) it follows that

$$x(t) = x_0\{y_1(t=0)x_1(t)-y_2(t=0)x_2(t)\}$$

$$- y_0\sqrt{k_a/k_b}\{x_1(t=0)x_2(t)-x_2(t=0)x_1(t)\}. \qquad (14)$$

The expression (14) is the generalization of the well-known form of the X force level, $x(t)$, for the classical Lanchester (constant coefficient) equations of "modern warfare" to the most general case of variable Lanchester attrition-rate coefficients. The reader should note the key role that (11) plays in verifying that (14) does indeed satisfy the initial conditions to (6). We also see from (14) that once results are obtained for one member of a family of battles, force levels may be readily deduced for other members of this family.

Recalling the well-known constant coefficient results, the reader knows that for $t_0 \neq 0$ (14) is at least sometimes capable of further simplification. However, it is indeed remarkable that this is only true when (9) holds.

THEOREM 1: For $t_0 \neq 0$, any further simplification in (14) is possible <u>if and only if</u> (9) holds (constant ratio of attrition-rate coefficients).

PROOF: From (8) we have, for example, that

$$\frac{dx_1}{dy_2} = k^2 \frac{a(t)}{b(t)} \frac{y_2}{x_1} .$$

Thus a relationship that is independent of t exists between x_1 and y_2 if and only if $a(t)/b(t)$ = constant. Moreover, when such a relationship does exist, we have via (9) that

$$[x_1(t)]^2 - [y_2(t)]^2 = 1. \qquad (15)$$

Besides being a classical Lanchester "square law," equation (15) is a necessary and sufficient condition, for example, for $x_1(t)$ to possess a so-called algebraic addition theorem (see Note 6). Unless the general Lanchester functions possess such algebraic addition theorems, for $t_0 \neq 0$ there is no further simplification to (14).

Q.E.D.

Remark 1: When (9) does hold, we have that $x_1(t) = y_1(t)$ and $x_2(t) = y_2(t)$, where $x_1(t)$ and $y_2(t)$ are given by (10), so that the complementary X- and Y-functions are (only in this case) identical. Moreover, (11) (or, equivalently, (15) is then a well-known property of the hyperbolic functions. Furthermore, in this case (14) becomes the well-known result [11], [29], [34], [35]

$$x(t) = x_0 \cosh(\sqrt{k_a k_b} \int_0^t h(s)ds) - y_0 \sqrt{k_a/k_b} \sinh(\sqrt{k_a k_b} \int_0^t h(s)ds), \qquad (16)$$

thanks to the well-known (algebraic) addition theorems for the hyperbolic functions, for example, like cosh u cosh v - sinh u sinh v = cosh (u-v). It is also enlightening to observe that (16) may be written as

$$x(t) = x_0 \cosh(\sqrt{\overline{a(t)b(t)}}\ t) - y_0 \sqrt{a(t)/b(t)} \sinh(\sqrt{\overline{a(t)b(t)}}\ t), \qquad (17)$$

where $\sqrt{\overline{a(t)b(t)}}$ denotes an average, i.e. $\sqrt{\overline{a(t)b(t)}} = \frac{1}{t}\int_0^t \sqrt{a(s)b(s)}\, ds$. Thus, we see that in the case of a constant ratio of attrition-rate coefficients, the X force level, x(t), is given by an expression formally equivalent to that for the constant coefficient case with averages being used.

Remark 2: In cases in which a (square-law) relationship like (15) does not hold, there is no such convenient reduction of (14) with $t_0 \neq 0$ to a simpler form like (16) via an algebraic addition theorem. Thus, in general the X force level, x(t), does not take a simple form except when $t_0 = 0$ or $a(t)/b(t) =$ constant.

Remark 3: When $t_0 = 0$, then (14) again takes a particularly simple form

$$x(t) = x_0 x_1(t) - y_0 \sqrt{k_a/k_b}\, x_2(t). \tag{17}$$

Our previous results [37] were all of the form (17).

In a previous paper [37] (see also [34], [35]) we showed that when the ratio of attrition-rate coefficients is constant, we can transform the X force-level equation into one with constant coefficients by a transformation of the independent variable t. As we have seen, this case leads to particularly convenient results. In this respect, a useful theorem is

THEOREM 2: A necessary and sufficient condition to be able to transform the X force-level equation (6) by a transformation of the independent variable t into a linear second order ordinary differential equation with constant coefficients is that

$$\frac{1}{\sqrt{a(t)b(t)}} \left\{ \frac{1}{a(t)} \frac{da}{dt} - \frac{1}{b(t)} \frac{db}{dt} \right\} = \text{CONSTANT}.$$

PROOF: The theorem follows immediately from a result given on pp. 73-74 of [4]. Q.E.D. Moreover, when the X force-level equation is so transformable, the desired sutstitution is given by $u = K \int^t \sqrt{a(s)b(s)}\, ds$, where $\int^t \ldots ds$ denotes an indefinite integral and K is an arbitrary constant conveniently chosen.

For example, for the power attrition rate coefficients (4), Theorem 2 tells us that we can transform (6) into an equation with constant coefficients only when (I) $m = n$, or (II) $m + n + 2 = 0$.

5. Power Attrition-Rate Coefficients With Both Systems' Effectiveness Zero at the Same Time.

In this case the attrition-rate coefficients are given by (4). There are two cases to be considered, depending on whether (6) with (4) can be transformed into a constant coefficient equation (see Theorem 2 above):

(I) $m + n + 2 \neq 0$,

and (II) $m + n + 2 = 0$,

Case I. $m + n + 2 \neq 0$:

In order for solution methods (either successive approximations or the method of Frobenius) to be applicable, we must further impose the following restrictions: (a) for $C = 0$, we must have $m > -1$ and $n > -1$, while (b) for $C > 0$, we must have $m + n + 2 \neq 0$. It should be noted that only the former case was considered in our previous paper [37] and that the theory presented in the last section is essential for extending these results to the latter case.

From results given in [37], it follows that a fundamental system of solutions to (6) with attrition-rate coefficients (4) is given by

$$x_1(t) = \Gamma(q)(\sqrt{k_a k_b}/2s)^p (t+C)^{(m+1)/2} I_{-p}(\sqrt{k_a k_b}(t+C)^s/s), \tag{18}$$

and

$$x_2(t) = \Gamma(p)(\sqrt{k_a k_b}/2s)^q (t+C)^{(m+1)/2} I_p(\sqrt{k_a k_b}(t+C)^s/s), \tag{19}$$

where I_p denotes the modified Bessel function of the first kind of order p, $p = (m+1)/(m+n+2)$, $p + q = 1$, and $2s = m + n + 2$, with similar results holding for $\{y_1(t), y_2(t)\}$. The above Lanchester functions, of course, satisfy all the properties given in Table I. The X force level, $x(t)$, then is given by(14).

To emphasize the dependence on the parameter C it is sometimes convenient to write, for example, $x_1(t) = x_1(t;C)$. It should be noted then that $x_1(t=0;C=0) = y_1(t=0;C=0) = 1$ and $x_2(t=0;C=0) = y_2(t=0;C=0) = 0$ so that (14) with these particular fundamental systems reduces to (17) when $C = 0$. It should be noted that no such simplification of (14) occurs for $C > 0$ unless $m = n$. In other words, the parameter t_0 with the property (13) is given by $t_0 = -C$.

The above representations (18) and (19) of fundamental systems of solutions to the X and Y force-level equations are not particularly convenient because, for

example, for $m,n > -1$ we have $0 < p,q$ and $p + q = 1$, and tabulations only exist of the modified Bessel function of the first kind of fractional order ν for a restrictive set of values (i.e. $\nu = \pm 1/4, \pm 1/3, \pm 2/3, \pm 3/4$ (see [37])). This Bessel function reduces to other tabulated forms (i.e. hyperbolic functions) for $\nu = \pm 1/2$. Therefore, it is more useful to express $x_1(t)$, $x_2(t)$, $y_1(t)$, and $y_2(t)$ in the form of infinite series and to consider the resulting transcendental functions as entities in their own right. Thus, it is convenient to define the folloiwng Lanchester functions

$$u_{m,n}(t) = \Gamma(q) \sum_{k=0}^{\infty} \left(\frac{\sqrt{k_a k_b}}{m+n+2}\right)^{2k} \frac{t^{k(m+n+2)}}{k!\Gamma(k+q)}, \tag{20}$$

$$v_{m,n}(t) = \Gamma(p) \sum_{k=0}^{\infty} \left(\frac{\sqrt{k_a k_b}}{m+n+2}\right)^{2k+1} \frac{t^{k(m+n+2)+m+1}}{k!\Gamma(k+1+p)}, \tag{21}$$

$$U_{m,n}(t) = \Gamma(p) \sum_{k=0}^{\infty} \left(\frac{\sqrt{k_a k_b}}{m+n+2}\right)^{2k} \frac{t^{k(m+n+2)}}{k!\Gamma(k+p)}, \tag{22}$$

$$V_{m,n}(t) = \Gamma(q) \sum_{k=0}^{\infty} \left(\frac{\sqrt{k_a k_b}}{m+n+2}\right)^{2k+1} \frac{t^{k(m+n+2)+n+1}}{k!\Gamma(k+1+q)}. \tag{23}$$

The Lanchester functions $u_{m,n}$, $v_{m,n}$, $U_{m,n}$, and $V_{m,n}$ have the properties shown in Table II. For reasons to be explained below we will refer to $u_{m,n}(t)$ and $U_{m,n}(t)$ as complementary Lanchester functions and similarly for $v_{m,n}(t)$ and $V_{m,n}(t)$. The solution (14) to (6) may then be written in terms of these Lanchester functions as

$$x(t) = x_0\{U_{m,n}(C)u_{m,n}(t+C) - V_{m,n}(C)v_{m,n}(t+C)\}$$

$$- y_0\sqrt{k_a/k_b}\{u_{m,n}(C)v_{m,n}(t+C) - u_{m,n}(t+C)v_{m,n}(C)\}, \tag{24}$$

and similarly for $y(t)$. From (24) we see that the methods of [37] would become hopelessly bogged down in details for $C > 0$.

For computational reasons it is convenient to introduce the auxiliary Lanchester functions (also referred to as the Lanchester-Clifford-Schläfli functions, or LCS functions (see Note 7)), which are defined for $\nu \neq 0, -1, -2, -3, \ldots$

Table II. Properties of the Lanchester Functions $u_{m,n}$, $v_{m,n}$, $U_{m,n}$, and $V_{m,n}$.

1. $$\frac{du_{m,n}}{dt} = \sqrt{k_b/k_a}\ [k_a t^m] V_{m,n}(t)$$

2. $$\frac{dv_{m,n}}{dt} = \sqrt{k_b/k_a}\ [k_a t^m] U_{m,n}(t)$$

3. $$\frac{dU_{m,n}}{dt} = \sqrt{k_a/k_b}\ [k_b t^n] v_{m,n}(t)$$

4. $$\frac{dV_{m,n}}{dt} = \sqrt{k_a/k_b}\ [k_b t^n] u_{m,n}(t)$$

5. $$u_{m,n}(t)U_{m,n}(t) - v_{m,n}(t)V_{m,n}(t) = 1 \qquad \forall t$$

6. $$u_{m,n}(t=0) = U_{m,n}(t=0) = 1$$

7. $$v_{m,n}(t=0) = V_{m,n}(t=0) = 0$$

8. $$u_{m,m}(t) = U_{m,m}(t) = \cosh\left(\sqrt{k_a k_b}\ t^{m+1}/(m+1)\right)$$

9. $$v_{m,m}(t) = V_{m,m}(t) = \sinh\ \sqrt{k_a k_b}\ t^{m+1}/(m+1))$$

$$F_\nu(x) = \sum_{k=0}^{\infty} (x/2)^{2k}/\{k! \prod_{j=0}^{k-1} (j+\nu)\}, \tag{25}$$

and

$$G_\nu(x) = \sum_{k=0}^{\infty} (x/2)^{2k+1}/\{k! \prod_{j=0}^{k} (j+\nu)\}, \tag{26}$$

where we have adopted the convention that $\prod_{j=M}^{N} f_j = 1$ for $N < M$. The LCS functions possess the properties shown in Table III.

The Lanchester functions $u_{m,n}$, $v_{m,n}$, $U_{m,n}$, and $V_{m,n}$ may be expressed in terms of the LCS functions as follows

$$u_{m,n}(t) = F_q(\psi(t)), \tag{27}$$

$$v_{m,n}(t) = t^{(m-n)/2} G_p(\psi(t)), \tag{28}$$

$$U_{m,n}(t) = F_p(\psi(t)), \tag{29}$$

$$V_{m,n}(t) = t^{(n-m)/2} G_q(\psi(t)), \tag{30}$$

where $\psi(t) = \sqrt{[k_a t^m][k_b t^n]}\ t/((m+n+2)/2)$. Thus, $u_{m,n}(t)$ and $U_{m,n}(t)$ have been called complementary Lanchester functions because of the above relationships, the fact that $p + q = 1$, and property 3. of Table III. Additionally, the introduction of the LCS functions sheds light on the parametric dependence of solutions: there are two exponent parameters, p and $(m-n)/2$, and an "intensity" parameter, $\theta(t) = \sqrt{a(t)b(t)}\ (t+C)/((m+n+2)/2)$, as well as the relative effectiveness parameter $\sqrt{k_a/k_b}$. We finally note that (24) may now be written as

$$x(t) = x_0\{F_p(\theta(0))F_q(\theta(t)) - (1+t/C)^{(m-n)/2} G_q(\theta(0))G_p(\theta(t))\}$$

$$- y_0\sqrt{a(t=0)/b(t=0)}\{(1+t/C)^{(m-n)/2} F_q(\theta(0))G_p(\theta(t)) - G_p(\theta(0))F_q(\theta(t))\}, \tag{31}$$

where $\theta(t) = \sqrt{a(t)b(t)}\ (t+C)/((m+n+2)/2)$. It should be observed that average "intensity" $\overline{\sqrt{a(t)b(t)}}$ is related to $\theta(t) = \theta(t;C)$ by

$$\overline{\sqrt{a(t)b(t)}}\ t = \theta(t;C) - \theta(t=0;C).$$

Table III. Properties of the LCS Functions $F_\nu(x)$ and $G_\nu(x)$.

1. $$\frac{dF_\nu}{dx} = G_\nu(x)$$

2. $$\frac{dG_\nu}{dx} = F_\nu(x) - (2\nu-1)\frac{G_\nu(x)}{x}$$

3. $$F_p(x)F_q(x) - G_p(x)G_q(x) = 1 \qquad \forall t$$
   where $p + q = 1$ and $p,q \neq 0,-1,-2,-3,\ldots$

4. $$F_\nu(x=0) = 1$$

5. $$G_\nu(x=0) = 0$$

6. $$\lim_{x\to 0} \frac{G_\nu(x)}{x} = \frac{1}{2\nu}$$

7. $$\frac{dF_\nu}{dx}(x=0) = 0$$

8. $$\frac{dG_\nu}{dx}(x=0) = \frac{1}{2\nu}$$

9. $$F_{1/2}(x) = \cosh x$$

10. $$G_{1/2}(x) = \sinh x$$

Without an algebraic addition theorem, however, this does not lead to simplification of (31). We do find, though, that for $C = 0$ with $m,n > -1$ we have $\sqrt{a(t)b(t)}\ t = \theta(t;C=0)$, and (31) may be written like (17) as

$$x(t) = x_0\ F_q(\sqrt{a(t)b(t)}\ t) - y_0\sqrt{a(t)/b(t)}\ G_p(\sqrt{a(t)b(t)}\ t). \qquad (32)$$

It may be shown that $\lim_{t\to 0} \sqrt{a(t)/b(t)} \cdot G_p(\sqrt{a(t)b(t)}\ t) = 0$.

Considering (31) and the above, there is an interesting way (with some similarities with the constant coefficient situation) of thinking about combat between two homogeneous forces described by (1) and (4): average combat "intensity" as well as how this changes over time determines the course of combat besides the initial force levels, x_0 and y_0, and $\sqrt{a(t=0)/b(t=0)}$.

Case II. $m + n + 2 = 0$ and $C > 0$:

It should first be noted that the solution obtained for Case I becomes indeterminate (consider, for example, what happens to $u_{m,n}(t)$ as defined by (20)). To solve (6) with (4) and $m + n + 2 = 0$, we make the substitution $u = \ell n\ t$ to transform (6) into an equation with constant coefficients. Solving this equation, we find that one fundamental system (with the properties given in Table I) for (14) is given by

$$x_1(t) = -\frac{C^{m+1}}{\sqrt{2\theta}}\sqrt{\frac{\beta_-}{\alpha_-}}\left(\frac{t+C}{C}\right)^{\alpha_-},$$

$$x_2(t) = \frac{\beta_+}{\sqrt{2\theta}}\left(\frac{t+C}{C}\right)^{\alpha_+},$$

and similarly for $\{y_1(t), y_2(t)\}$, where $\theta = \sqrt{k_a k_b + ((m+1)/2)^2}$, $\alpha_+ = \theta + (m+1)/2 = -\beta_-$, and $\alpha_- = -\theta + (m+1)/2 = -\beta_+$. In this case (14) becomes

$$x(t) = x_0\left\{\frac{1}{2\theta}\left[\beta_+\left(\frac{t+C}{C}\right)^{\alpha_+} - \beta_-\left(\frac{t+C}{C}\right)^{\alpha_-}\right]\right\}$$

$$- y_0\sqrt{\frac{k_a}{k_b}}\left\{\frac{\sqrt{k_a k_b}\ C^{m+1}}{2\theta}\left[\left(\frac{t+C}{C}\right)^{\alpha_+} - \left(\frac{t+C}{C}\right)^{\alpha_-}\right]\right\}, \qquad (33)$$

which may also be written in the more convenient form

$$x(t) = \left(\frac{t+C}{C}\right)^{(m+1)/2}\left\{x_0\left[\cosh\left(\theta\ \ell n\left[\frac{t+C}{C}\right]\right)-\left(\frac{m+1}{2\theta}\right)\sinh\left(\theta\left[\frac{t+C}{C}\right]\right)\right]\right.$$

$$\left. - y_0\sqrt{\frac{k_a}{k_b}}\left[\frac{C^{m+1}\sqrt{k_a k_b}}{\theta}\sinh\left(\theta\ \ell n\left[\frac{t+C}{C}\right]\right)\right]\right\}. \qquad (34)$$

Although the above solutions appear complex, they are readily evaluated with the help of a ("hand-sized") portable electronic calculator such as is commercially available today. (In fact, such a calculator can even be "programmed" to facilitate parametric analyses.)

6. Weapon Systems With Different Effective Ranges: Linear Attrition-Rate Coefficients.

Another situation of interest in which to apply our general theory is that of combat between two homogeneous forces which use weapons with different effective ranges. Let us consider the example previously examined by us [37](see also [11]) of a constant speed attack of a mobile force against a static defense. The weapon systems of the two sides have different effective ranges, and the lethality of each side's fire depends linearly upon range. We assume that the opening range of battle, R_0, is $\leq$ minimum (R_α, R_β), where R_α denotes the maximum effective range of the Y system. The Lanchester attrition-rate coefficients for such a battle may be written as (5). The parameter C is related to the opening range of battle in comparison with minimum (R_α, R_β), whereas A reflects the difference in maximum effective ranges. The range dependencies of these Lanchester attrition-rate coefficients and the opening range of battle is shown in Figure 1.

From results given in [37], it follows that a fundamental system of solutions to (6) (and also one for the Y force-level equation) with the attrition-rate coefficients (5) (which has the properties shown in Table I) is given by

$$x_1(t) = f(t+C), \quad x_2(t) = g(t+C), \quad y_1(t) = F(t+C), \quad \text{and} \quad y_2(t) = G(t+C), \qquad (35)$$

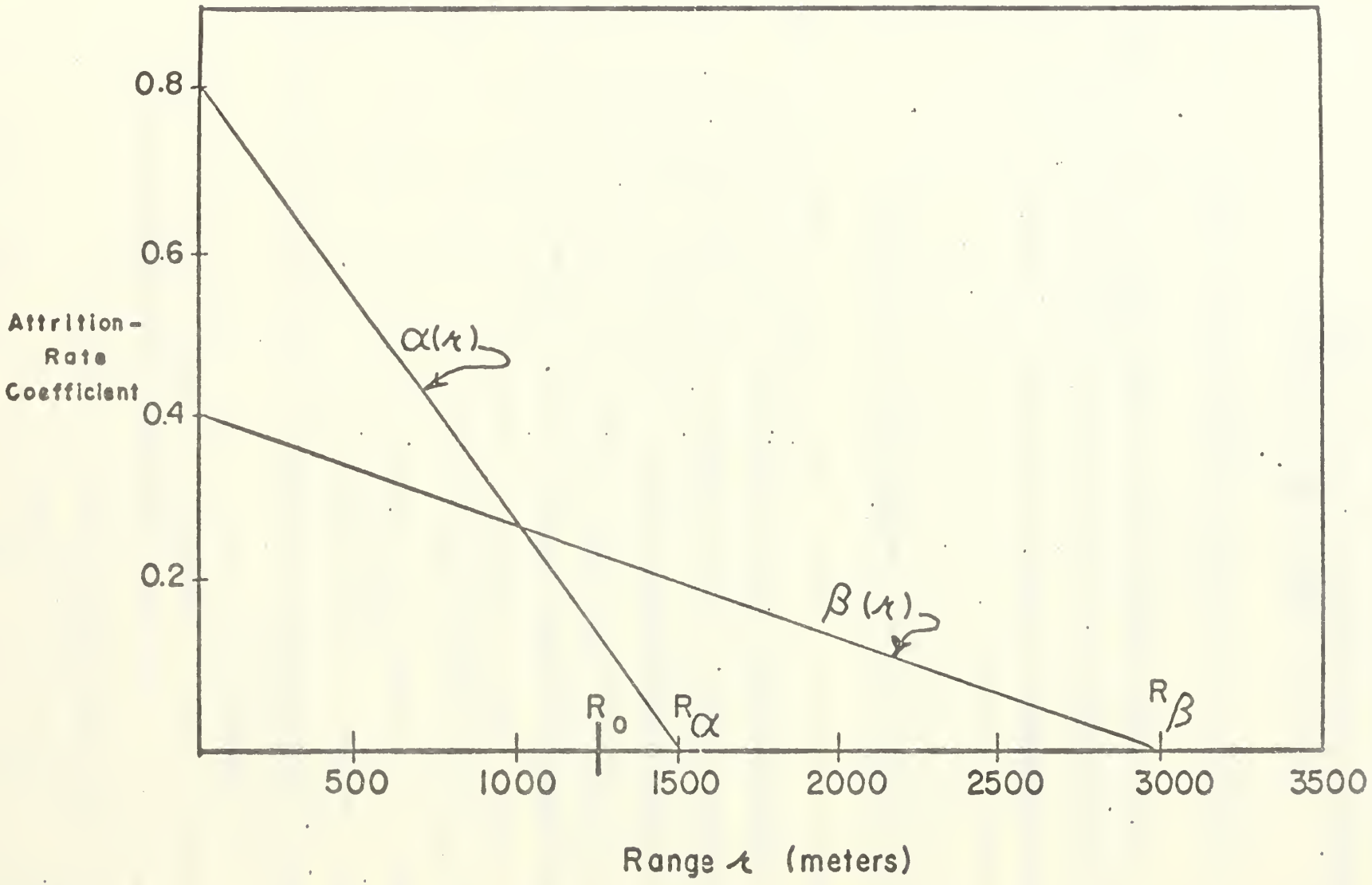


Figure 1. Linear attrition-rate coefficients for weapon systems with different effective ranges. [Notes: 1. The maximum effective ranges of the two weapon systems are denoted as R_α and R_β. 2. The opening range of battle is denoted as R_0 and (as shown) $R_0 <$ minimum (R_α, R_β).]

where

$$f(t) = \sum_{n=0}^{\infty} \frac{(\sqrt{k_a k_b}/2)^{2n}}{(2n)!} \sum_{k=0}^{n} B_n^k A^k t^{4n-k}, \tag{36}$$

$$g(t) = \sum_{n=0}^{\infty} \frac{(\sqrt{k_a k_b}/2)^{2n+1}}{(2n+1)!} \sum_{k=0}^{n} C_n^k A^k t^{4n+2-k}, \tag{37}$$

$$F(t) = \sum_{n=0}^{\infty} \frac{(\sqrt{k_a k_b}/2)^{2n}}{(2n)!} \sum_{k=0}^{n} D_n^k A^k t^{4n-k}, \tag{38}$$

$$G(t) = \sum_{n=0}^{\infty} \frac{(\sqrt{k_a k_b}/2)^{2n+1}}{(2n+1)!} \sum_{k=0}^{n+1} E_n^k A^k t^{4n+2-k}, \tag{39}$$

and the coefficients B_n^k and C_n^k are given in [37]. The coefficients D_n^k and E_n^k are given by $D_0^0 = 1$ and for $n > 0$

$$D_n^k = \begin{cases} 1, & \text{for } k = 0, \\ [4n(4n-2)/(4n-k)][D_{n-1}^k/(4n-k-2)) + (D_{n-1}^{k-1}/(4n-k-1))], & \text{for } 1 \le k \le n-1, \\ (4/3)[(4n-2)/(3n-1)]D_{n-1}^{n-1}, & \text{for } k = n; \end{cases}$$

and $E_0^0 = 1$, $E_0^1 = 2$, and for $n > 0$

$$E_n^k = \begin{cases} 1, & \text{for } k = 0, \\ [4n(4n+2)/(4n-k+2)][(E_{n-1}^k/(4n-k)) + (E_{n-1}^{k-1}/(4n-k+1))], & \text{for } 1 \le k \le n, \\ (4/3)[(4n+2)/(3n+1)]E_{n-1}^n, & \text{for } k = n+1. \end{cases}$$

It should be noted that $f(t=0) = F(t=0) = 1$ and $g(t=0) = G(t=0) = 0$.

To emphasize the dependence on the parameter A, it is sometimes convenient to write, for example, $f(t) = f(t;A)$. It should be noted that $f(t;A=0) = F(t;A=0) = \cosh(\sqrt{k_a k_b}\, t^2/2)$ and $g(t;A=0) = G(t;A=0) = \sinh(\sqrt{k_a k_b}\, t^2/2)$. For $A = 0$, the fundamental property $f(t)F(t) - g(t)G(t) = 1$ of these Lanchester functions becomes a well-known property of the hyperbolic functions.

As was the case for the power attrition-rate coefficients of the previous section, for computational reasons it is convenient to introduce the following auxiliary Lanchester functions (depending on two parameters)

$$h(\lambda,\mu) = \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!} \sum_{k=0}^{n} B_n^k \mu^k, \tag{40}$$

$$w(\lambda,\mu) = \sum_{n=0}^{\infty} \frac{\lambda^{2n+1}}{(2n+1)!} \sum_{k=0}^{n} C_n^k \mu^k, \tag{41}$$

$$H(\lambda,\mu) = \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!} \sum_{k=0}^{n} D_n^k \mu^k, \tag{42}$$

$$W(\lambda,\mu) = \sum_{n=0}^{\infty} \frac{\lambda^{2n+1}}{(2n+1)!} \sum_{k=0}^{n+1} E_n^k \mu^k. \tag{43}$$

These auxiliary Lanchester functions possess the following properties:

1. $h(\lambda,\mu)H(\lambda,\mu) - w(\lambda,\mu)W(\lambda,\mu) = 1 \qquad \forall \lambda,\mu$

2. $h(\lambda,0) = H(\lambda,0) = \cosh \lambda,$

2. $w(\lambda,0) = W(\lambda,0) = \sinh \lambda.$

Using the offset linear auxiliary Lanchester functions, the X force level may be written as

$$x(t) = x_0\{H(\theta(0),\delta(0))h(\theta(t),\delta(t)) - W(\theta(0),\delta(0))w(\theta(t),\delta(t))\}$$

$$- y_0\sqrt{k_a/k_b}\{h(\theta(0),\delta(0))w(\theta(t),\delta(t)) - w(\theta(0),\delta(0))h(\theta(t),\delta(t))\}, \tag{44}$$

where $\delta(t) = A/(t+C)$ and $\theta(t) = \sqrt{k_a k_b}\,(t+C)^2/2$. The expression $\delta(t)$ reflects how much the above Lanchester functions deviate from the hyperbolic functions.

7. Some Numerical Examples.

In this section we examine three numerical examples which illustrate possible use of some of our new results. These examples are extensions of the ones given previously in [37] by having the opening range of battle be less than the minimum of the two maximum effective ranges for the two weapon systems. They are motivated by the work of S. Bonder [6], [8], [11] on the value of range capabilities and mobility for weapon systems in combat described by Lanchester-type equations of modern warfare.

The modelling context of these examples is that of weapon systems with (a) different range dependencies of lethality of each side's fire (but the same maximum effective range) and (b) linear attrition-rate coefficients but different effective ranges.

As in [37] we consider a constant-speed attack on a static defensive position with the combat dynamics described by

$$dx/dt = -\alpha(r)y = -\alpha_0(1-r/R_\alpha)^m y, \qquad dy/dt = -\beta(r)x = -\beta_0(1-r/R_\beta)^n x, \qquad (45)$$

where R_α and R_β denote the maximum effective ranges of the Y and X weapon systems, respectively (i.e. $\alpha(r) = 0$ for $r > R_\alpha$). Range is related to time by $r(t) = R_0 - vt$, where R_0 denotes the opening range of battle. Several range dependencies for an attrition-rate coefficient are shown in Figure 2, and an opening range less than the weapon system's maximum effective range is indicated. The parameters of the attrition-rate coefficients $\alpha(r(t))$ and $\beta(r(t))$ in (45) are readily related to those of $a(t)$ and $b(t)$ in (4) and (5). For example, we have for (4) that $k_a = \alpha_0(v/R_\alpha)^m$, $k_b = \beta_0(v/R_\beta)^n$, and $C = (R_\alpha - R_0)/v$, where $R_\alpha = R_\beta$.

Numerical results are shown in Figures 3, 4, and 5. The force-level trajectories have been generated by a digital computer program using the auxiliary Lanchester functions (i.e. F_ν and G_ν for (31) and h, w, H, and W for (44)). These functions are particularly convenient for such digital computer work. The computer routines were checked against the numerical results given previously in [37]. In this work we have taken the opening range of battle $R_0 <$ minimum (R_α, R_β). Numerical values for battle parameters (except those for R_0, R_α, and R_β) are the same as those used for the examples considered in [37].

For Figures 3 and 4 both weapon systems have the same maximum effective range (i.e. $R_\alpha = R_\beta$). As done for the plots in [37], we have held $\alpha_0 = \alpha(r=0)$ and β_0 constant and have varied the exponents m and n which control the range dependencies of $\alpha(r)$ and $\beta(r)$. With the exception of R_0, all the battle parameters for the

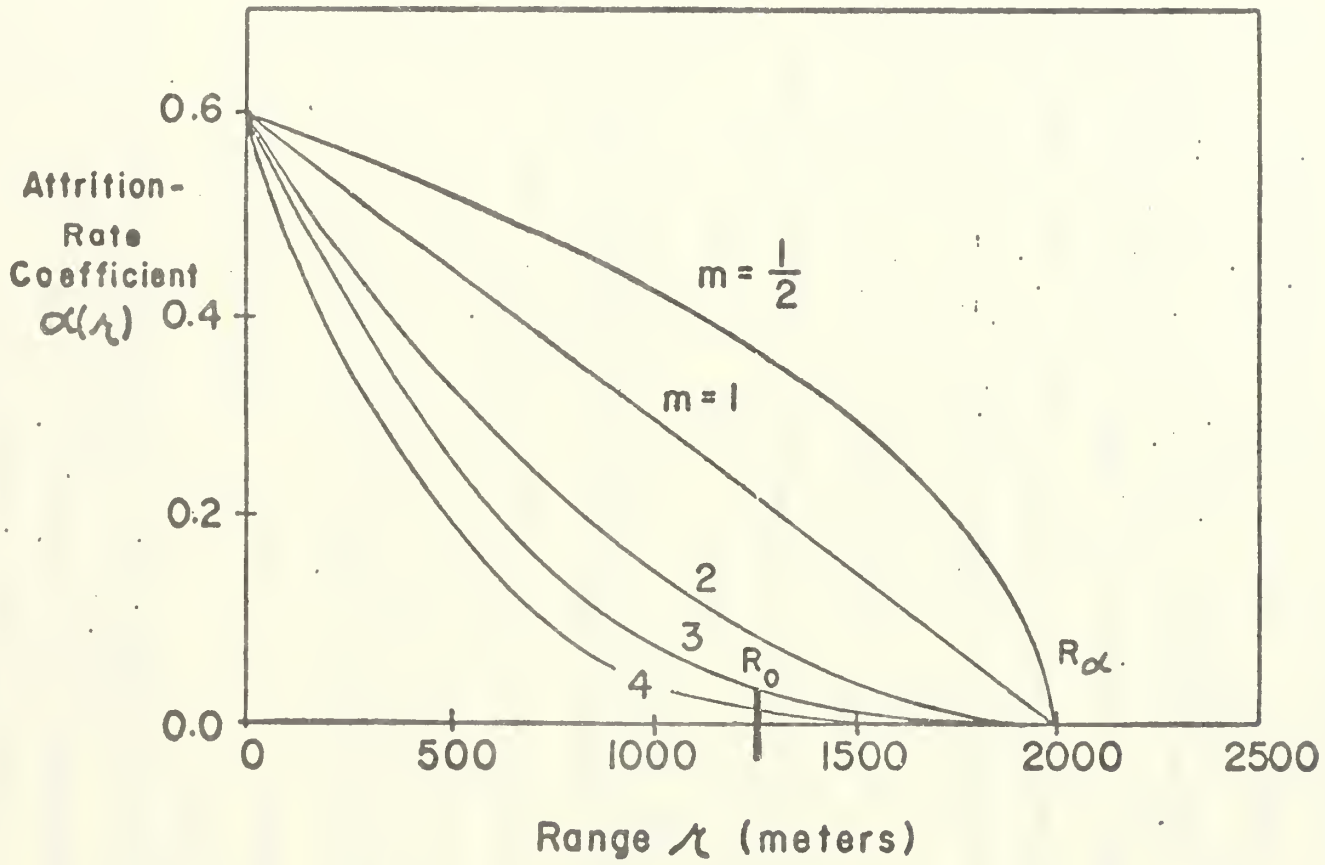


Figure 2. Dependence of the attrition-rate coefficient $\alpha(r)$ on the exponent m for constant maximum effective range of the weapon system and constant kill capability at zero range. [Notes: 1. The maximum effective range of the system is denoted R_α = 2000 meters. 2. $\alpha(r=0) = \alpha_0$ = 0.6 X casualties/(unit time × number of Y units) denotes the Y force weapon system kill rate at zero force separation (range). 3. The opening range of battle is denoted as R_0 = 1250 meters and (as shown) $R_0 < R_\alpha$.]

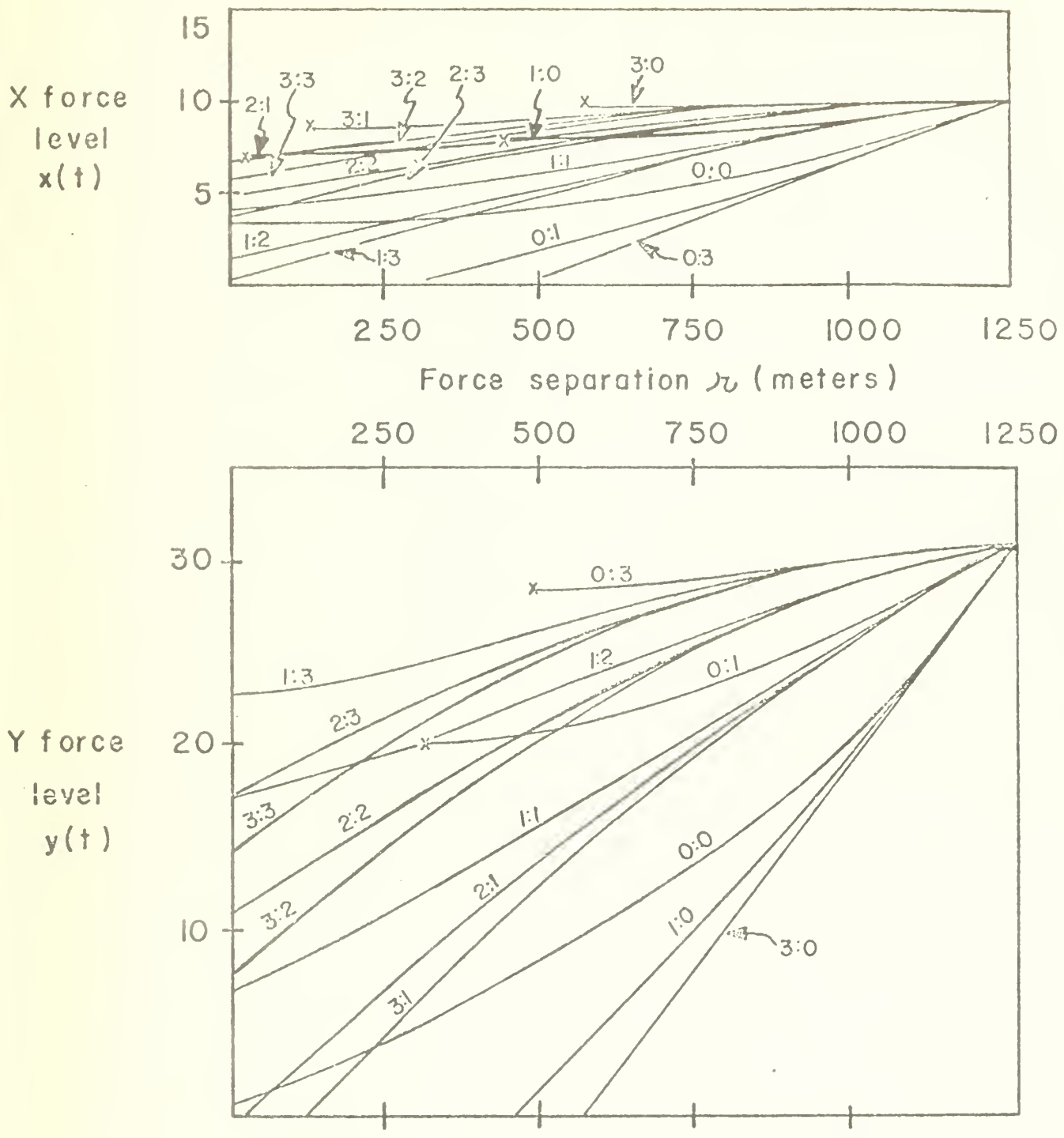


Figure 3. Force-level trajectories of X and Y forces for various combinations of the exponents m and n in the power attrition-rate coefficients for R_0 = 1250 meters, $R_\alpha = R_\beta$ = 2000 meters, α_0 = 0.06 X casualties/(minutes · Y unit), β_0 = 0.6 Y casualties/(minutes · X unit), v = 5 mph, x_0 = 10, and y_0 = 30. The exponent combinations are denoted as m:n in the figure, and the symbol × denotes the end of a force-level trajectory due to the annihilation of the enemy force.

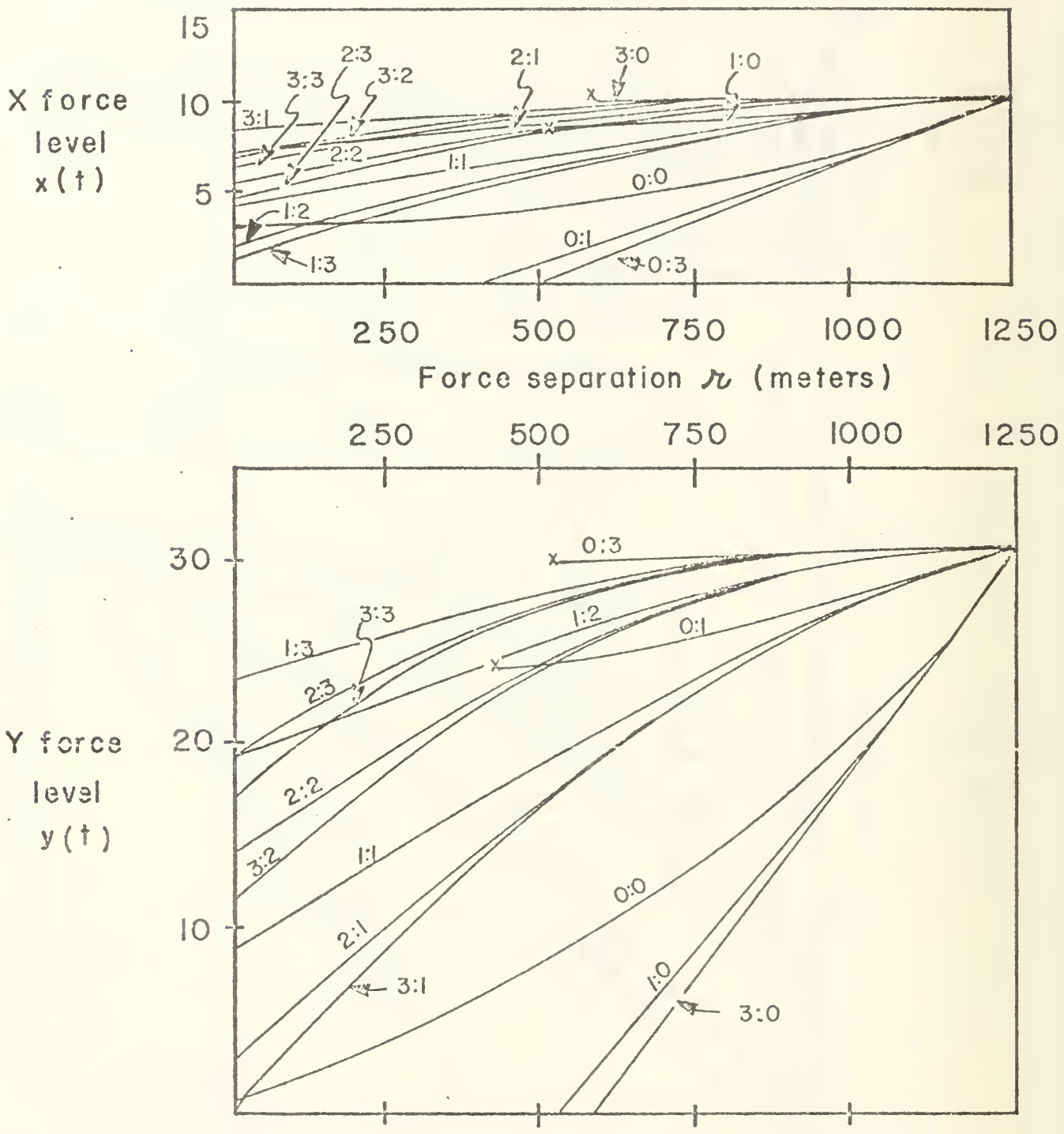
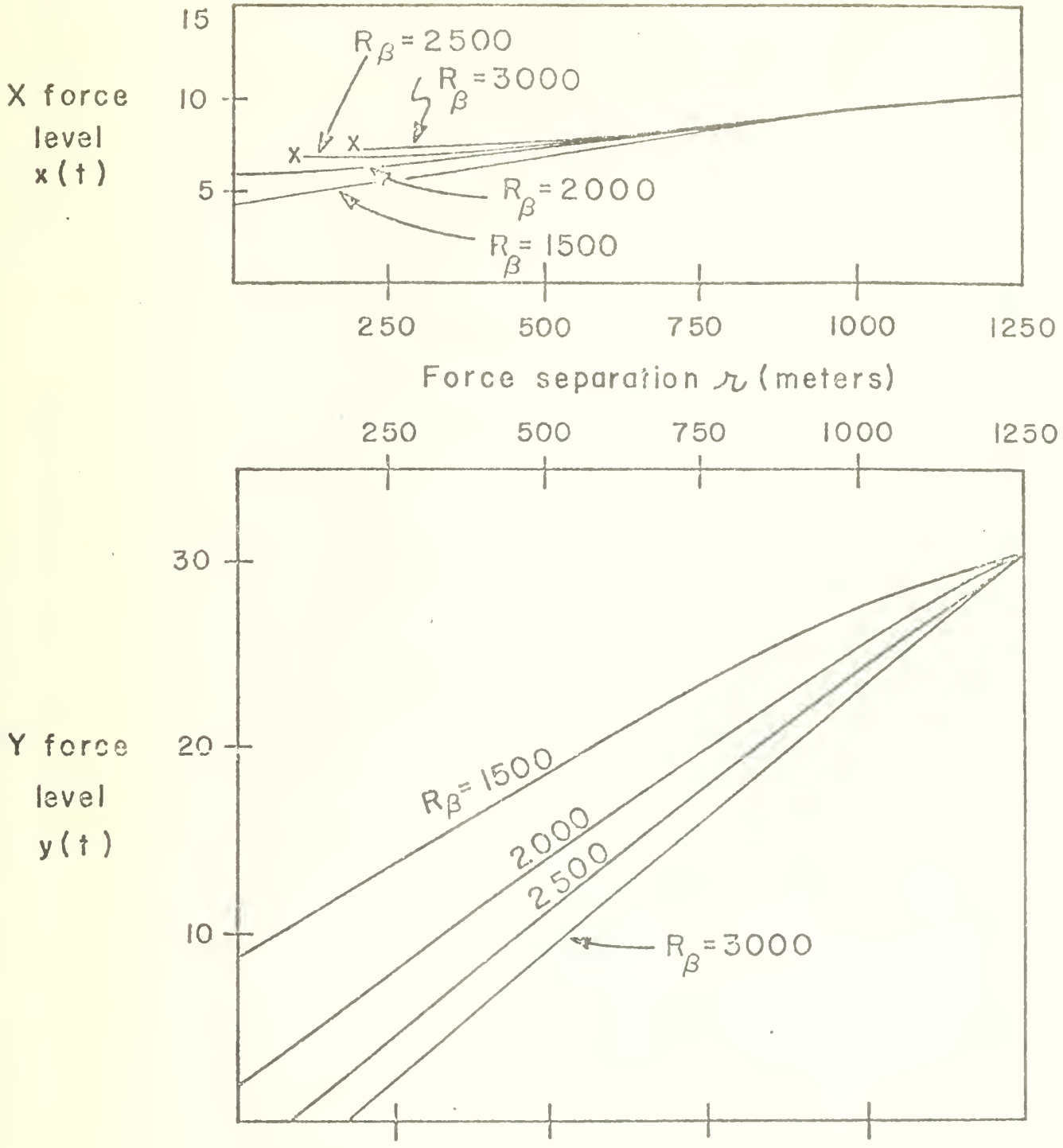


Figure 4. Force-level trajectories of X and Y forces for various combinations of the exponents m and n in the power attrition-rate coefficients for the same parameter values chosen for Fig. 3 except that $R_\alpha = R_\beta = 1500$ meters. The symbol conventions are also the same as in Fig. 3.



igure 5. Force-level trajectories of X and Y forces for various effective ranges R_β of the X force weapons with linear attrition-rate coefficients for R_α = 1500 meters and the same values of the other parameters listed in the legend of Fig. 3. The symbol × has the same meaning as in this figure.

curves shown in Figure 3 are the same as those used for the corresponding example previously considered in [37] (i.e. $R_\alpha = R_\beta = 2000$ meters). Consequently, corresponding force-level trajectories are similar with greater "separation" shown here between curves. In Figure 3 with the opening range of battle $R_0 = 1250$ meters, the curves corresponding to the constant-coefficient case (i.e. $m = n = 0$) are exactly the same (for the same time intervals) as those shown in [37] with $R_0 = 2000$ meters. Other battle trajectories with $m,n > 0$ decay faster in Figure 3 than they did in [37] because the "intensity" of combat is greater (i.e. as a function of time the attrition-rate coefficients are larger here than they were in [37] (see Figure 2)).

As noted in [37], knowledge of the range- (or time-) dependence of weapon system kill capability is essential for forecasting the battle's outcome from the initial trend of battle. For example, compare the outcomes for curves denoted as 1:0, 1:1, and 1:2. G. Clark [13] has developed methodology for estimating such capability from the output of a (high resolution) Monte Carlo combat simulation. The "compounding" nature of attrition over time for (45) is evident from the curves shown in Figure 3: a small advantage in numbers and firepower becomes magnified over time. Similar battle curves are shown in Figure 4 for the same parameter values except that $R_\alpha = R_\beta = 1500$ meters. Observing that for $m \geq 1$ we have $\alpha(r;R_\alpha) < \alpha(r;\bar{R}_\alpha) \Leftrightarrow R_\alpha < \bar{R}_\alpha$, we may consider that the combat is less "intense" for such pairs of battle trajectories in Figure 4 than for those shown in Figure 3.

In Figure 5 we show the effect of increasing the effective range of the defende weapons. [The X force may be considered to be the defender.] For these computation (using (44) and the auxiliary Lanchester functions) we have held the opening range of battle constant at $R_0 = 1250$ meters and the maximum effective range of the Y weapon constant at $R_\alpha = 1500$ meters. As in [37], both attrition-rate coefficients depend linearly on range (i.e. $m = n = 1$ in (45)), α_0 and β_0 have been held constant, and R_β has been varied. As shown in Figure 5, we quantitatively see the benefit

from increasing the long-range kill capability of the defender's weapons. (See pp. 296-367 of [11] for extensive parametric studies based on analogue computer results. Our analytic results given in the paper at hand may be used for such investigations.) It should be emphasized that although the results shown in Figures 3, 4, and 5 are similar to those in [37], our previous analytic results were limited to the case in which the opening range of battle R_0 = minimum (R_α, R_β).

8. The Force-Ratio Equation.

As the above results show, the analytic solution of the variable-coefficient Lanchester-type equations of modern warfare (1) provides little insight into the prediction of outcome without explicitly generating force-level trajectories (see Figure 3 through 5) because of its complexity (nor does numerical solution (for a specific set of conditions) by finite difference methods (see, for example, [41])). We can gain, however, some qualitative insights into the dynamics of combat between two homogeneous forces by considering the equation satisfied by the force ratio $u = x/y$:

$$du/dt = b(t)u^2 - a(t), \tag{46}$$

where $u(t=0) = u_0 = x_0/y_0$ (see [40] (also [11], [22])). There is no analytic advantag in trying to solve the (generalized) Riccati equation (46), since the standard method of solution (see [40]) results in transforming (46) into (6). If we consider a battle which terminates at the first time that either of two given "breakpoint" force ratios (denoted as u_X^f when X wins and as u_Y^f when Y wins and satisfying $0 \leq u_Y^f < u_0 < u_X^f \leq +\infty$), then we may say that "the course of battle is moving towards an X victory" when $du/dt > 0$. Thus, we obtain the "instantaneous" square law

$$b(t)x^2(t) > a(t)y^2(t), \tag{47}$$

which may be considered to be a "local" condition for X to win. The "instantaneous" square law (47) holds even when $a(t)/b(t)$ is not constant and there is no relationship between x and y that is independent of t (like the classical Lanchester

square law (2)). Moreover, if $R(t) = a(t)/b(t)$ is a nonincreasing function of time, then (47) holding at $t = 0$ is a sufficient condition for X to "win" (or at least to improve the force ratio in his favor) (see [40]).

As discussed in [40], for the model (1) the advantage to X from concentration of forces is seen by considering the instantaneous (or differential) casualty exchange ratio

$$dx/dy = a(t)/\{b(t)u\}, \tag{48}$$

which represents the cost to X of reducing the enemy (Y) force level by a unit amount. In this case, it is always to X's advantage to make x_0 as large as possible since $\partial(dx/dy)/\partial u < 0$ (see [40]). If supporting fires are considered, however, this will not always be true. Thus, in our nuclear age, one must tradeoff the advantages of concentration of forces against vulnerability to supporting fires (see [40]). It should be emphasized that the above qualitative behavior of the solution to the force-ratio equation (46) has been inferred without explicitly solving (46).

9. Extensions.

There are two directions in which the results of the paper at hand may be extend (a) other attrition-rate coefficients and (b) other combat dynamics. The first direction of extension to other functional forms of attrition-rate coefficients is straightforward but may prove to be messy. The authors are not too optimistic about extension to other combat dynamics (even if we limit ourselves to "square-law" attrition processe (see [36], [39])). Figure 6 (after some ideas of one of the founders of "systems theory," Ludwig von Bertalanffy (1901-1972) (see pp. 20-21 of [5])) contains our thoughts on the subject. A digital computer and finite difference methods, however, allow one to generate numerical particular solutions to combat equations that are essentially unvolvable analytically.

In Figure 6 we take the Lanchester-type equations of modern warfare for combat between two homogeneous forces with supporting fires not subject to attrition (see [40] for further discussion of this model) to be given by (for $x,y > 0$)

	No Replacements		Replacements	
	Constant Coefficients	Variable Coefficients	Constant Coefficients	Variable Coefficients
Two Homogeneous Forces	*Very Easy*	*Difficult*	*Easy*	*Very Difficult*
Two Homogeneous Forces With Supporting Fires Not Subject to Attrition	*Easy*	*Very Difficult*	*Not Too Easy*	*Very Difficult*
Heterogeneous Forces (Several Combatant Types)	*Difficult*	*Essentially Impossible*	*Very Difficult*	*Impossible*
Heterogeneous Forces (Many Combatant Types)	*Essentially Impossible*	*Impossible*	*Impossible*	*Impossible*

Figure 6. Classification of Lanchester-Type Equations of "Modern Warfare" and Their Ease of Solution by Analytical Methods (after L. Bertalanffy [5]).

$$dx/dt = -a(t)y - \beta(t)x \qquad \text{with} \quad x(t=0) = x_0,$$
$$dy/dt = -b(t)x - \alpha(t)y \qquad \text{with} \quad y(t=0) = y_0, \tag{49}$$

where the attrition-rate coefficients are, of course, nonnegative. In this case the X force-level equation is given by

$$\frac{d^2x}{dt^2} + \left(\alpha(t)+\beta(t) - \frac{1}{a(t)}\frac{da}{dt}\right)\frac{dx}{dt} + \left(\alpha(t)\beta(t)+a(t)\frac{d}{dt}\left\{\frac{\beta(t)}{a(t)}\right\} - a(t)b(t)\right)x = 0. \tag{50}$$

The theory of this paper (also that of [37]) may be applied to (49) (or equivalently (50)). However, to us this appears to represent essentially the limits of fruitful analytic investigation. Except for special cases, the equations for combat between heterogeneous forces do not appear to be amenable to analytic solution. To be sure, one can write down "symbolic solutions" (such as the matrix exponential for constant attrition-rate coefficients (see [11], [37])), but these are apparently not computationally useful.

10. Discussion.

In this section we discuss the significance and applications of the results of this paper. These results may be used to facilitate parametric analysis of the dynamic combat interactions between two homogeneous forces with time- (or range-) dependent weapon system capabilities. Such models are of particular interest in light of the work of S. Bonder [7], [9] and others [3], [11], [31], [32] on the prediction of Lanchester attrition-rate coefficients from weapon system performance data and the work of G. Clark [13] on the estimation of such (time-dependent) coefficients from Monte Carlo simulation output. Additionally, our new theoretical results may be used to solve Grubbs and Shuford's [19] new probabilistic formulation for Lanchester combat theory (see equation (3)). A further discussion of applications is to be found in [37] (see also [11]).

We have presented a general mathematical theory for the solution of variable coefficient Lanchester-type equations of "modern warfare" for combat between two

homogeneous forces. These results allow one to extend results given in [37] that applied under rather restrictive conditions (e.g. opening range of battle equal to the minimum of the two maximum effective ranges of the weapon systems) (see also [11]). It has been shown that in general the deterministic time histories of the X and Y force levels (i.e. x(t) and y(t)) may be expressed in terms of four complementary Lanchester functions (and not two as in the constant-coefficient case). The mathematical properties of these general Lanchester functions that facilitate analytic solution have been given. We have given particular attention to the determination of the conditions on attrition-rate coefficients that allow relatively simple analytic results.

Motivated by the analytic solutions of such equations for certain functional forms of attrition-rate coefficients of interest (i.e. power attrition-rate coefficients and linear attrition-rate coefficients with different effective ranges), we have proposed new mathematical functions (and given their mathematical properties) which could be tabulated in the future. This would allow analysts to generate numerical results for such variable attrition-rate coefficient combat formulations with somewhat the same facility as one can for the constant-coefficient case and thus aid in parametric analyses. This means that numerical results for a single battle can be used to generate numerical solutions for an entire family of battles. Such results for the reference battle need not even be generated by analytic means but could be developed, for example, by numerical integration. Thus one can combine the general theoretical results given here with tabulations generated by finite difference methods. We have discussed what appear to be the limitations of such analytic investigations as far as complexity of combat dynamics.

The results of the paper at hand and [37] hopefully provide the theoretical foundations for the generation of force-level trajectories via "analytic" solutions for variable-coefficient combat formulations (Lanchester-type equations of "modern

warfare") with approximately the same degree of thoroughness in these foundations as previously existed for the constant-coefficient case. In other words, we have stressed computational procedures here. For qualitative insights into the dynamics of combat via the force ratio reference should be made to the recent paper by Taylor and Parry [40].

11. Notes.

1. This is because of the complexity of obtaining analytic results from stochastic formulations. The computational limitations of Monte Carlo simulation (especially high resolution or for combat between large units) are well known. The work of G. Clark shows (see pp. 102-103 of [13]) the complexity of an analytic solution for even the most idealized combat situation modelled as a continuous parameter Markov chain (see [13] for further references). Moreover, Bonder and Farrell [11] report excellent agreement between simulation results and those for a corresponding deterministic Lanchester-type model (numerical solution generated by finite difference methods (see [41])). In this sense modern high speed digital computers have facilitated the application of general systems theory (see [5], especially pp. 17-20) to military systems. Of course, verification of such models (as with any combat model) is an unresolved question (see [10]; further references are given in [37]).

2. Although finite-difference mehtods (see [41]) and a modern high speed digital computer can generate approximate numerical solutions to Lanchester-type equations with theoretically any degree of desired accuracy, it nevertheless is of interest to have analytic solutions available (see [28]), if for no other reason than to be able to check the adequacy of finite-difference approximation. Considering the ease of generating such numerical solutions, it is curious that Barfoot states that (p. 888 of [3]) "existing Lanchester models of combat have assumed constant attrition-rate coefficients for the sake of simplicity." This statement, in fact, apparently goes back (via Bonder (p. 231 of [7])) to Dolansky (p. 345 of [16]), who also stressed the (at that

time, 1964) lack of valid means for determining such coefficients. As one of the authors has shown [34], for a constant ratio of attrition-rate coefficients the solution to such variable-coefficient equations for combat between two homogeneous forces is no more complicated than that for constant coefficients.

3. The influential military philosopher of the 19th century, Carl von Clausewitz (1780-1831) stated in his classic work On War (Vom Kriege) (p. 276 of [14]), "The best Strategy is always to be very strong, first generally then at the decisive point. ...There is no more imperative and no simpler law for Strategy than to keep the forces concentrated."

4. There is, moreover, far from universal agreement as to what are the significant variables which describe the combat process and can be used to predict its outcome. For some other views see [21] and [26].

5. The differential equation under consideration could not be found among the 445 linear second order equations tabulated in [24] or the 596 tabulated in [30].

6. For a discussion of algebraic addition theorems see Chapter II of [20]. Harris Hancock gives the following theorem (see p. 37 of [20]): a necessary and sufficient condition for a single-valued analytic function $f(z)$ to possess an algebraic addition theorem is that there exist between the function $f(z)$ and its first derivative an algebraic equation whose coefficients are independent of the argument z. No more comprehensive result could be found in the more recent monograph by Aczel [2].

7. A function similar to $F_\nu(x)$ was introduced by Ludwig Schläfli in 1867 [33] and another appears in a posthumous fragment by William Kingdon Clifford (1845-1879) (see pp. 346-348 of [15]). Greenhill [18] suggested that such a function would be convenient for certain engineering-like problems, although he was severely criticized by

Watson (see p. 91 of [42]). Since tabulations of none of the above or modified Bessel functions of the first kind of fractional order (except for the restrictive set of values $\nu = \pm 1/4, \pm 1/3, \pm 1/2, \pm 2/3, \pm 3/4$) exist, it would seem appropriate to introduce the LCS functions as we have done.

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